Geometry Lecture Notes

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1 Affine and projective spaces

1.1 Affine spaces

Definition *Affine space*

Let E be a vector space over a field $\mathbb{K}$, and $\mathcal{E}$ a set. Then $\mathcal{E}$ is an **affine space** if there exists a map

$$f : \mathcal{E} \times \mathcal{E} \to E \qquad (A, B) \mapsto \overrightarrow{AB}$$

such that:

1. for all $A, B \in \mathcal{E}$, the partial map $f(A, \cdot)$ is a bijection from $\mathcal{E}$ to E.
2. for all $A, B, C \in \mathcal{E}$, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Lemma

Let $\mathcal{E}$ be an affine space. Then we have for all $A, B \in \mathcal{E}$:

1. $\overrightarrow{AA} = 0$
2. $\overrightarrow{AB} = -\overrightarrow{BA}$
3. $f(\cdot, B)$ is a bijection from $\mathcal{E}$ to E

Definition *Adding points and vectors*

Let $A, B \in \mathcal{E}$ and $\vec{v} \in E$ such that $\vec{v} = \overrightarrow{AB}$. Then $A + \vec{v} = B$.

Proposition

The point B from the previous definition exists and is unique.

Proposition

Every vector space is also an affine space, where $\overrightarrow{AB} = \vec{A} - \vec{B}$.

Proposition *Parallelogram rule*

$$\overrightarrow{AB} = \overrightarrow{CD} \implies \overrightarrow{AC} = \overrightarrow{BD}$$

1.1.1 Affine subspaces

Definition *Affine subspace*

A subset $\mathcal{F}$ of an affine space $\mathcal{E}$ is an **affine subspace** if there exists a point $A \in \mathcal{F}$ such that

$$F = \{\overrightarrow{AB}, B \in \mathcal{F}\}$$

is a vector subspace of E. The subspace $\mathcal{F}$ is said to be **directed** by F.

Lemma

If there exists A such that the previous definition holds, then the definition holds for all A.

Proposition

Let $A \in \mathcal{E}$ and let F be a vector subspace of E.
Then there is a unique affine subspace of $\mathcal{E}$ directed by F and passing through A.

Definition *Line, 2-plane*

Affine subspaces of dimension 1 are called **lines**, affine subspaces of dimension 2 are called **2-planes**. (note: dimension is understood as the dimension of the underlying vector space)

Definition *Parallel subspaces*

Two lines L_1, L_2 in an affine space $\mathcal{E}$ are **parallel** if they have the same direction.
Similarly, two affine subspaces $\mathcal{F}_1, \mathcal{F}_2$ are parallel if they are directed by the same vector space.

Proposition

For any line L in an affine space $\mathcal{E}$ and a point $A \notin L$ in $\mathcal{E}$, there is a unique line parallel to L passing through A.

1.1.2 Affine transformations

Definition *Affine transformation*

Let $\mathcal{E}$ and $\mathcal{F}$ be affine spaces directed by vector spaces E and F respectively.
A map $f : \mathcal{E} \to \mathcal{F}$ is an **affine transformation** if

$$\text{there exists } A \in \mathcal{E} \text{ such that for all } B \in \mathcal{E},\ \overrightarrow{AB} \mapsto \vec{f}(\overrightarrow{AB}) = \overrightarrow{f(A)f(B)} \text{ is a linear map } E \to F$$

Lemma

If there exists $A \in \mathcal{E}$ such that the previous definition holds, then it holds for all $A \in \mathcal{E}$.

Proposition

All affine transformations are of the form $f(x) = T(x) + b$, where T is a linear map and b a point.

Theorem *Thales' theorem*

Let $\mathcal{D}_1, \mathcal{D}_2$ be two lines in an affine space $\mathcal{E}$ and d, d', d'' parallel lines, and let $A_1, A_1', A_1'', A_2, A_2', A_2''$ be their respective intersection points. Then

$$\frac{\overrightarrow{A_1A_1''}}{\overrightarrow{A_1A_1'}} = \frac{\overrightarrow{A_2A_2''}}{\overrightarrow{A_2A_2'}}$$

Conversely, if $B \in \mathcal{D}_1$ is such that

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1A_1'}} = \frac{\overrightarrow{A_2A_2''}}{\overrightarrow{A_2A_2'}}$$

then $B = A_1''$.

Theorem *Fundamental theorem of affine geometry*

Let $\mathcal{E}, \mathcal{F}$ be affine spaces of dimension ≥ 2.
If $\varphi : \mathcal{E} \to \mathcal{F}$ is a bijection such that for all triples (A, B, C) of points in $\mathcal{E}$ the images $\varphi(A), \varphi(B), \varphi(C)$ are colinear (i.e. on the same line), then φ is an affine mapping, and moreover an isomorphism.

1.2 Euclidean geometry

Definition *Euclidean geometry*

Euclidean geometry is the affine geometry on $\mathbb{R}^n$, equipped with the following inner product:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

We denote the Euclidean space by $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$.

Definition *Distance*

In Euclidean space, we define the **distance** between points by

$$d(x, y) = \sqrt{\langle x - y, x - y \rangle}$$

Note

On the exam, there will be fewer questions about Euclidean geometry, compared to other topics.
Reflections in circles and/or lines will not be on the exam.

1.3 Projective spaces

Definition *Projective space*

Let E be a finite-dimensional vector space. The **projective space** $P(E)$ deduced from E is the set of all lines in E. The dimension of this space is $\dim E - 1$. We can denote $P(\mathbb{R}^n)$ by $\mathbb{R}P^{n-1}$.

Proposition

Every line in $P(\mathbb{R}^2)$ can be uniquely represented by a point on the half unit circle, or a point in $L \cup \{\infty\}$. Here, L is any horizontal line not intersecting the origin, and L is isomorphic to $\mathbb{R}$.

2 Curves

Examples of curves

Brachistochrone: $\begin{cases} x = r(\varphi - \sin\varphi) + c \\ y = r(1 - \cos\phi) \end{cases}$ **Bernoulli's lemniscate**: $\begin{cases} x = \dfrac{t}{1+t^4} \cdot d \\ y = \dfrac{t^3}{1+t^4} \cdot d \end{cases}$

2.1 Smooth and regular curves

Definition *Curve*

A curve is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$.
A curve is called **smooth** if each component of the curve is infinitely differentiable.
A curve is called **regular** if $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Definition *Singular point*

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth curve. $t \in [a, b]$ is a **singular point** if $\gamma'(t) = 0$.

Note

Singular points are often isolated, but this need not be so, and cusps are not the only examples.

2.2 Envelopes

Definition *Tangent line*

Let $\mathbb{R}^2$ be an affine and Euclidean space. The **tangent line** to f at (x_0, y_0) is:

$$y = y_0 + f'(x_0)(x - x_0)$$

Definition *Envelope*

Let $\{D_t\}$ be a family of lines in $\mathbb{R}^2$. (not necessarily through the origin)
Then the **envelope** to D_t is a curve $\gamma : [a, b] \to \mathbb{R}^2$ such that for all $t \in [a, b]$:

- $\gamma'(t)$ is parallel to D_t
- $\gamma(t) \in D_t$

Computation of envelopes

Let $\{D_t\}$ be a family of lines defined by $u(t)x + v(t)y + w(t) = 0$.
Then we can compute the envelope $\gamma = (x(t), y(t))$ by solving the following linear system:

$$\begin{cases} u(t)x(t) + v(t)y(t) + w(t) = 0 \\ u'(t)x(t) + v'(t)y(t) + w'(t) = 0 \end{cases}$$

We can write this as a matrix-vector equation:

$$\underbrace{\begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix}}_{A} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ w'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This equation can be solved by inverting A. There exists a unique envelope if $\det A \neq 0$ for all $t \in [a, b]$. Solving the matrix-vector equation will lead to the following expressions for x and y:

$$x = -\frac{wv' - w'v}{uv' - vu'} \qquad y = -\frac{w'u - wu'}{uv' - vu'}$$

2.3 Curvature

Definition *Arc length*

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1-smooth curve. The **arc-length parameter** on γ is defined as

$$s = s(t) := \int_a^t \left| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right| \mathrm{d}t$$

Proposition

Assume that γ is a smooth and regular curve. Then

- $s = s(t)$ is a smooth, strictly monotone function of t.
- The inverse $t(s)$ of $s(t)$ is also smooth and strictly monotone.
- $|\gamma'(s)| = 1$

Definition *Unit tangent and normal vector*

The **unit tangent vector** $\tau(s)$ is equal to the derivative of the arc length parametrization $\gamma(s)$.
The **unit normal vector** $n(s)$ is chosen such that (τ, n) is a positively oriented orthonormal basis.

Definition *Curvature*

For a regular curve $\gamma : [a, b] \to \mathbb{R}^3$, let s be its arc length parameter and let $n(s)$ be the normal vector to γ at s. If

$$\tau'(s) = \kappa(s) \cdot n(s)$$

then $\kappa(s)$ is the **curvature** of γ at s.

Theorem *Frenet formulas in $\mathbb{R}^2$*

$$\begin{bmatrix} \tau'(s) \\ n'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \begin{bmatrix} \tau(s) \\ n(s) \end{bmatrix}$$

Definition *Curvature radius and center*

The **curvature radius** $\rho(s)$ is defined by

$$\rho(s) = \frac{1}{\kappa(s)}$$

and the **curvature center** $C(s)$ is defined by

$$C(s) = \gamma(s) + \rho(s) \cdot n(s)$$

The **osculating circle** at t is the circle at $C(s)$ with radius $|\rho(s)|$.

2.4 Evolutes

Definition *Evolute*

The **evolute** of a regular C^2 curve $\gamma : [a, b] \to \mathbb{R}^2$ is equal to the set of curvature centers of γ.

Proposition

If $D(t)$ is the family of normal lines to a curve γ, then the envelope of $D(t)$ is the evolute of γ.

Definition *Wavefronts*

Let γ be a curve with unit normal $n(t)$. **Wavefronts** of γ are parallel curves of the form

$$\gamma_a(t) = \gamma(t) + an(t)$$

Theorem *Huygens' principle*

$\gamma_a(t)$ is singular (i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_a(t) = 0$) at points where $\rho(t) = a$.
The set of all of these singular points (at all a) is the evolute of γ.

3 Surfaces

Definition *Surface*

A **surface** $S \subset \mathbb{R}^n$ is defined by a set of parametric equations:

$$\begin{cases} x_1 = f_1(u_1, \ldots, u_{n-1}) \\ \vdots \\ x_n = f_n(u_1, \ldots, u_{n-1}) \end{cases}$$

where f is C^1 and (usually) its Jacobian is surjective for all $(u_1, \ldots, u_{n-1}) \in \mathbb{R}^{n-1}$.

A surface can also be defined by an **implicit equation**:

$$S = \{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) = 0\}$$

where f is C^1 and its gradient is nowhere zero.

3.1 Ruled and revolution surfaces

Definition *Ruled surface*

Let γ be a curve and $w(p)$ a vector defined at each point p on γ. Let D_p denote the line going through each point $p \in \gamma([a, b])$, directed by $w(p)$. The union of all lines D_p is a **ruled surface**.

Examples of ruled surfaces

A **cylinder** is a ruled surface where the vector $w(p)$ is constant.
A **cone** is a ruled surface where $w(p) = \vec{Op}$ for some fixed point O.

Definition *Surface of revolution*

Consider a plane curve C and make it turn about a line D of its plane.
The line D is called the **axis of revolution**, and the surface that is produced this way is a **surface of revolution**.

Examples of surfaces of revolution

If the curve C is a straight line, the surface obtained is a **cylinder** if the lines C and D are parallel, a **plane** if they are perpendicular, and a **cone** otherwise.

If the curve C is a circle, the surface obtained is a **sphere** if the line D is a diameter of C, or a **torus** if D and C do not intersect.

Möbius strip

$$\gamma(r,\phi) = (\cos\phi + r\cos(\phi/2),\, \sin\phi,\, t\sin(\phi/2)) \qquad r \in [-1,1], \phi \in [0, 2\pi]$$

3.2 Quadrics

Definition *Quadric*

A **quadric** in $\mathbb{R}^n$ is the locus of a degree 2 polynomial:

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid p(x) = 0\} \qquad p(x_1, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + c$$

Definition *Classification of quadrics*

Regular curves in dimension $n = 2$:

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parabola: $y = \frac{x^2}{a}$

Singular sets in dimension $n = 2$:

Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

Singular surfaces in dimension $n = 3$:

Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

Cylinder over 1-cone: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Regular surfaces in dimension $n = 3$:

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

1-sheeted hyperboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

2-sheeted hyperboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Hyperbolic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Elliptic cylinder: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbolic cylinder: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Parabolic cylinder: $y = \frac{x^2}{a}$

Other quadrics are the empty set, a point, a line, a plane, a pair of points, a pair of lines, and a pair of planes.
For images of these quadrics, see Week 3, Lecture 7 on Brightspace.

3.3 Tangent spaces and fundamental forms

Definition *Tangent space*

Let $M \subset \mathbb{R}^n$ be a regular surface.
Then the **tangent space** T_pM of M at a point $p \in M$ can be described in one of the following equivalent ways:

1. Let $\vec{n}(p)$ be a normal vector to M at p.
   Then T_pM is the plane in $\mathbb{R}^n$ orthogonal to $\vec{n}(p)$ and passing through p.
2. $$T_pM = \{\gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^n \text{ is a } C^1\text{-smooth curve with } \gamma(0) = p\}$$

Definition *Gauss map*

Let M be a surface in $\mathbb{R}^3$ and S^2 the unit sphere of $\mathbb{R}^3$.
The map $\vec{n} : M \to S^2$, mapping a point to the unit normal vector at that point, is called the **Gauss map**.

Proposition

Let $M \subset \mathbb{R}^3$ be a surface and S^2 the unit sphere. Then

$$T_pM \cong T_{\vec{n}(p)}S^2$$

Definition *Fundamental forms*

Let $M \subset \mathbb{R}^3$ be a surface, $p \in M$ a point, and X, Y vectors in T_pM.
The **first fundamental form**, denoted $I_p(X, Y)$, is the restriction of the Euclidean inner product $\langle X, Y \rangle$ to T_pM.
The first fundamental form is also called the **Riemannian metric**.
The **second fundamental form** is defined by

$$II_p(X, Y) = -\langle \mathrm{d}\vec{n}(x), Y \rangle = -I_p(\mathrm{d}\vec{n}(x), Y)$$

Fundamental forms (matrix form)

Let $\vec{r}_1(p), \ldots, \vec{r}_{n-1}(p)$ be the natural basis of a surface M. The entries of the **matrix of the second fundamental form** are then given by

$$Q_{ij}(p) = II(\vec{r}_i(p), \vec{r}_j(p)) = \left\langle \frac{\partial^2}{\partial u_i \partial u_j} \vec{r}(p), \vec{r}(p) \right\rangle$$

Similarly, we have the **matrix of the first fundamental form**:

$$G_{ij}(p) = \langle \vec{r}_i(p), \vec{r}_j(p) \rangle$$

Proposition

The matrix of the second fundamental form is symmetric.

3.4 Gaussian and mean curvature

Definition *Gaussian and mean curvature*

Let $M \subset \mathbb{R}^3$ be a surface and $p \in M$ a point.
We define the **Gaussian curvature** at p by

$$k(p) = \det \mathrm{d}\vec{n}(p)$$

and the **mean curvature** at p by

$$H(p) = \mathrm{tr}\ \mathrm{d}\vec{n}(p)$$

Proposition

The Gaussian curvature $\kappa(p)$ satisfies

$$\kappa(p) = (-1)^{n-1} \frac{\det(Q_{ij}(p))}{\det(G_{ij}(p))}$$

Theorem *Gauss' Theorema Egregium*

The Gaussian curvature only depends on the Riemannian metric

Proposition

1. A plane in $\mathbb{R}^3$ has Gaussian and mean curvature 0.
2. A cylinder and cone have Gaussian curvature 0.
3. The disk S^2_R has constant Gaussian and mean curvature.
4. Helicoids and catenoids have negative Gaussian curvature and 0 mean curvature.
   (therefore they are minimal surfaces)

3.5 Isometries

Definition *Pullback*

$$f^*(G_N) = G_M \iff G_M(X,Y) = G_N(\mathrm{d}f(X), \mathrm{d}f(Y))$$

Definition *Local isometry*

Let $M, N \subset \mathbb{R}^3$ be surfaces. A **local isometry** is a smooth map $f : M \to N$ such that

$$f^*(G_N) = G_M$$

where G_M, G_N are the Riemannian metrics on M, N respectively, and $X, Y \in T_pM$.

Proposition

The Gaussian curvature is invariant under local isometries f:

$$\kappa_N \circ f \equiv \kappa_M$$

Definition *Classification of singular points*

Let $M \subset \mathbb{R}^3$ be a regular surface.
By applying an isometry, we can assume that for a given point $p \in M$, $\vec{n}(p) = (0,0,1)$.
If M around p is given by $z(x,y)$, with $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$, then the point p is

- **elliptic** if $\det[D^2 z(p)] > 0$
- **hyperbolic** if $\det[D^2 z(p)] < 0$
- **parabolic** if $\det[D^2 z(p)] = 0$ and $D^2 z(p) \neq 0$
- **planar** if $D^2 z(p) = 0$

Proposition

Let κ_p denote the Gaussian curvature at a point p.

$$\kappa_p > 0 \implies p \text{ elliptic} \qquad \kappa_p < 0 \implies p \text{ hyperbolic}$$

3.6 Geodesics

Definition *Riemannian distance*

Let g denote the Riemannian metric on $M \subset \mathbb{R}^3$. Then the **Riemannian distance** between $p, q \in M$ is

$$d(p,q) = \inf_{\mathcal{L}} \int_{t_1}^{t_2} \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\,\mathrm{d}t \quad \text{where} \quad \mathcal{L} = \{C^1\text{-smooth curves on } M \text{ such that } \gamma(t_1) = p, \gamma(t_2) = q\}$$

Definition *Geodesic*

A **geodesic** on $M \subset \mathbb{R}^3$ is a smooth curve $\gamma : I \to M$ parametrized by a constant multiple of arc-length, that locally realizes the distance function.

Proposition

Straight lines are geodesics of the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$.

3.7 Normal and principal curvature

Definition *Normal curvature*

Let $M \subset \mathbb{R}^3$ be a surface, let $X \in T_pM$ and let Π be the 2-plane in $\mathbb{R}^n$ spanned by the vectors X and $\vec{n}(p)$. The curvature K_x of the plane curve $\gamma = \Pi \cap M$ is called the **normal curvature** of M in the direction X.

Proposition

$$K_X = II_p(X/\|X\|, X/\|X\|)$$

Definition *Principal curvature*

Let $(e_1, \ldots, e_{n-1})$ be an orthonormal basis of T_pM such that the second fundamental form $Q_{ij}(p) = II_p(e_i, e_j)$ is diagonal. The diagonal entries K_{e_i} are called the **principal curvatures**.

Theorem *(Euler)*

Let $M \subset \mathbb{R}^3$ be a regular surface. Then the normal curvature can be expressed in terms of the principal curvatures:

$$K_X = \cos^2\theta K_{e_1} + \sin^2\theta K_{e_2} \qquad X = (r\cos\theta, r\sin\theta)$$

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